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FOURIER SERIES OF FUNCTIONS RELATED TO BERNOULLI POLYNOMIALS

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Abstract. We study the Fourier series of functions related to Bernoulli polynomials. As consequences, several new identities for the Bernoulli functions and numbers are derived.

1. Introduction

We know that Bernoulli numbers and polynomials appear everywhere in mathematics (for example, see [1, 3, 9–13]). The *Bernoulli numbers* have been defined by the generating function $\frac{t}{e^t-1} = \sum_{m\geq 0} B_m \frac{t^m}{m!}$. The *Bernoulli polynomials* $B_m(x)$ have been given by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{m\geq 0} B_m(x)\frac{t^m}{m!},$$

for any real number x, namely $x \in \mathbb{R}$. For instance, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x - 1/6$ and $B_3(x) = x^3 - 3x^2/2 + x/2$. For $u \in \mathbb{R}$, we denote the fractional part of u by $\langle u\rangle = u - \lfloor u \rfloor \in [0, 1)$. In this paper, we interested in three functions related to Bernoulli polynomials:

$$\alpha_m(x) = \sum_{k=0}^{m} B_k(x)x^{m-k}, \qquad \tilde{\alpha}_m(x) = \alpha_m(\langle x\rangle),$$

$$\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}B_k(x)x^{m-k}, \qquad \tilde{\beta}_m(x) = \beta_m(\langle x\rangle),$$

$$\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}B_k(x)x^{m-k}, \qquad \tilde{\gamma}_m(x) = \gamma_m(\langle x\rangle),$$

with $m \geq 1$, for $\alpha_m(x), \beta_m(x)$, and with $m \geq 2$, for $\gamma_m(x)$. We recall the following facts about Bernoulli functions:

$$-m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m} = \tilde{B}_m(x),\ m \geq 2, \tag{1}$$

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases} \tag{2}$$

where $\tilde{B}_m(x) = B_m(\langle x\rangle)$.

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The Fourier series of a periodic function $f(x)$ with period 1 is given by $\sum_{n=-\infty}^{\infty} f_n e^{2\pi inx}$, where the coefficients f_n are given by $f_n = \int_0^1 f(x)e^{-2\pi inx}dx$ (for example, see [2, 4, 9, 14–16]), where $i^2 = -1$.

The aim of this paper is to consider the *Fourier series* of $\tilde{\alpha}_m(x)$, $\tilde{\beta}_m(x)$ and $\tilde{\gamma}_m(x)$, which lead to several new identities for the Bernoulli functions and numbers.

2. The function $\tilde{\alpha}_m$

In this section, we consider the function $\tilde{\alpha}_m$ on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\tilde{\alpha}_m$ is $\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi inx}$, where

$$A_n^{(m)} = \int_0^1 \alpha_m(\langle x\rangle)e^{-2\pi inx}dx = \int_0^1 \alpha_m(x)e^{-2\pi inx}dx.$$

To proceed further, we note the following lemma.

Lemma 1. *For all $m \geq 1$, $\frac{d}{dx}\alpha_m(x) = (m+1)\alpha_{m-1}(x)$.*

Proof. By the definitions,

$$\begin{aligned}\frac{d}{dx}\alpha_m(x) &= \sum_{k=0}^{m}\left(kB_{k-1}(x)x^{m-k} + (m-k)B_k(x)x^{m-k-1}\right)\\ &= \sum_{k=0}^{m-1}\left((k+1)B_k(x)x^{m-1-k} + (m-k)B_k(x)x^{m-1-k}\right)\\ &= (m+1)\sum_{k=0}^{m-1} B_k(x)x^{m-1-k} = (m+1)\alpha_{m-1}(x),\end{aligned}$$

as claimed. □

By Lemma 1, we have $\int_0^1 \alpha_m(x)dx = \frac{\alpha_{m+1}(x)}{m+2}\Big|_{x=0}^{x=1} = \frac{\alpha_{m+1}(1)-\alpha_{m+1}(0)}{m+2}$. On the other hand

$$\alpha_m(1) - \alpha_m(0) = \sum_{k=0}^{m} B_k - (B_m - 1) = 1 + \sum_{k=0}^{m-1} B_k. \tag{3}$$

Thus,

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(1 + \sum_{k=0}^{m} B_k\right). \tag{4}$$

Now, we are ready to determine the Fourier coefficients $A_n^{(m)}$. First, let us consider the case $n \neq 0$. By Lemma 1 and (3), we obtain

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx = \frac{1}{2\pi i n} \int_0^1 \frac{d}{dx} \alpha_m(x) e^{-2\pi i n x} dx - \frac{1}{2\pi i n} \alpha_m(x) e^{-2\pi i n x} \Big|_{x=0}^{x=1} \\ &= \frac{m+1}{2\pi i n} \int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx - \frac{1}{2\pi i n} (\alpha_m(1) - \alpha_m(0)) \\ &= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n} \left(1 + \sum_{k=0}^{m-1} B_k \right). \end{aligned}$$

Hence, by induction on m, we obtain

$$A_n^{(m)} = \frac{(m+1)_{m-1}}{(2\pi i n)^{m-1}} A_n^{(1)} - \sum_{j=1}^{m-1} \frac{(m+1)_{j-1}}{(2\pi i n)^j} \left(1 + \sum_{k=0}^{m-j} B_k \right), \tag{5}$$

where $(x)_j = x(x-1)\cdots(x-j+1)$ with $(x)_0 = 1$. On the other hand, by the definitions, we have that $A_n^{(1)} = \int_0^1 \alpha_1(x) e^{-2\pi i n x} dx = \int_0^1 (2x - 1/2) e^{-2\pi i n x} dx = \frac{-1}{\pi i n}$, which, by (5), implies

$$A_n^{(m)} = -\frac{2(m+1)_{m-1}}{(2\pi i n)^m} - \sum_{j=1}^{m-1} \frac{(m+1)_{j-1}}{(2\pi i n)^j} \left(1 + \sum_{k=0}^{m-j} B_k \right).$$

Therefore,

$$A_n^{(m)} = -\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \left(1 + \sum_{k=0}^{m-j} B_k \right). \tag{6}$$

The case $n = 0$ follows immediately from (4):

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+2} \left(1 + \sum_{k=0}^{m} B_k \right). \tag{7}$$

Note that the function $\tilde{\alpha}_m$, $m \geq 1$, is piecewise C^∞. Moreover, the function $\tilde{\alpha}_m$ is continuous for those positive integers m with $\sum_{k=0}^{m-1} B_k = -1$, and discontinuous with jump discontinuities at integers for those positive integers m with $\sum_{k=0}^{m-1} B_k \neq -1$.

2.1. **Case $\sum_{k=0}^{m-1} B_k = -1$.** Assume first that m is a positive integer with $\sum_{k=0}^{m-1} B_k = -1$. Then $\alpha_m(1) = \alpha_m(0)$. So, the function $\tilde{\alpha}_m$ is piecewise C^∞ and continuous. Thus, the Fourier series of $\tilde{\alpha}_m$ converges uniformly to $\tilde{\alpha}_m$. So, by (6) an (7), we have

$$\tilde{\alpha}_m(x) = \frac{1}{m+2} \left(1 + \sum_{k=0}^{m} B_k \right) + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \left(1 + \sum_{k=0}^{m-j} B_k \right) \right) e^{2\pi i n x},$$

which, by (1) and (2), implies

$$\begin{aligned}\tilde{\alpha}_m(x) &= \frac{1}{m+2}\left(1+\sum_{k=0}^{m} B_k\right) + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\left(\left(1+\sum_{k=0}^{m-j} B_k\right)\sum_{n=-\infty,n\neq 0}^{\infty}\frac{-j!e^{2\pi i n x}}{(2\pi i n)^j}\right)\\ &= \frac{1}{m+2}\left(1+\sum_{k=0}^{m} B_k\right) + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\left(\left(1+\sum_{k=0}^{m-j} B_k\right)\tilde{B}_j(x)\right)\\ &+ \left(1+\sum_{k=0}^{m-1} B_k\right)\cdot\begin{cases}\tilde{B}_1(x), & x\notin\mathbb{Z},\\ 0, & x\in\mathbb{Z},\end{cases}\end{aligned}$$

which, by $\sum_{k=0}^{m-1} B_k = -1$, implies

$$\begin{aligned}\tilde{\alpha}_m(x) &= \frac{1}{m+2}\left(1+\sum_{k=0}^{m} B_k\right) + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\left(\left(1+\sum_{k=0}^{m-j} B_k\right)\tilde{B}_j(x)\right)\\ &= \frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\left(1+\sum_{k=0}^{m-j} B_k\right)\tilde{B}_j(x),\end{aligned}$$

for all $x \in \mathbb{R}$. Thus, we can state the following result.

Theorem 2. *Let m be a positive integer with $\sum_{k=0}^{m-1} B_k = -1$. Then the function $\tilde{\alpha}_m(x) = \sum_{k=0}^{m} \tilde{B}_k(x)\langle x\rangle^{m-k}$ has the Fourier series expansion*

$$\tilde{\alpha}_m(x) = \frac{1}{m+2}\left(1+\sum_{k=0}^{m} B_k\right) + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\left(1+\sum_{k=0}^{m-j} B_k\right)\right)e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the convergence is uniform. Moreover,

$$\tilde{\alpha}_m(x) = \frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\left(1+\sum_{k=0}^{m-j} B_k\right)\tilde{B}_j(x),$$

for all $x \in \mathbb{R}$.

2.2. **Case $\sum_{k=0}^{m-1} B_k \neq -1$.** Assume first that m is a positive integer with $\sum_{k=0}^{m-1} B_k \neq -1$. Then $\alpha_m(1) \neq \alpha_m(0)$. So, the function $\tilde{\alpha}_m$ is pointwise C^∞ and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $\tilde{\alpha}_m(x)$ converges piecewise to $\tilde{\alpha}_m(x)$ for all $x \notin \mathbb{Z}$, and converges to

$$\frac{\alpha_m(1)+\alpha_m(0)}{2} = \alpha_m(0) + \frac{1}{2}\left(1+\sum_{k=0}^{m-1} B_k\right), \quad \text{for all } x \in \mathbb{Z}.$$

Then, by Theorem 2, we obtain the following result.

Theorem 3. *Let m be a positive integer with $\sum_{k=0}^{m-1} B_k \neq -1$. Then*

$$\frac{1}{m+2}\left(1+\sum_{k=0}^{m} B_k\right)+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\left(1+\sum_{k=0}^{m-j} B_k\right)\right)e^{2\pi inx}$$
$$=\begin{cases}\tilde{\alpha}_m(x), & x\notin\mathbb{Z},\\ B_m+\frac{1}{2}\left(1+\sum_{k=0}^{m-1} B_k\right), & x\in\mathbb{Z}.\end{cases}$$

Moreover,

$$\frac{1}{m+2}\sum_{j=0}^{m}\binom{m+2}{j}\left(1+\sum_{k=0}^{m-j} B_k\right)\tilde{B}_j(x)=\tilde{\alpha}_m(x),\ \textit{for all } x\notin\mathbb{Z},$$

and

$$\frac{1}{m+2}\sum_{j=0,j\neq 1}^{m}\binom{m+2}{j}\left(1+\sum_{k=0}^{m-j} B_k\right)\tilde{B}_j(x)=B_m+\frac{1}{2}\left(1+\sum_{k=0}^{m-1} B_k\right),\ \textit{for all } x\in\mathbb{Z}.$$

Theorems 2 and 3 suggest the following question: For what values of integers $m \geq 1$ does $\sum_{k=0}^{m} B_k = -1$ hold?

We end this section by noting that the integral $\int_0^1 \alpha_m(x)dx = \sum_{k=0}^{m}\int_0^1 B_k(x)x^{m-k}dx$ has been obtained previously in [5–7] by determining $I_{p,q} = \int_0^1 B_p(x)x^q dx$, with $p, q \geq 0$. In fact, we can show that

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+1}+\sum_{k=1}^{m-1}\sum_{j=1}^{m-k}\frac{(-1)^{j-1}\binom{m-k+1}{j}B_{k+j}}{(m-k+1)\binom{k+j}{k}},$$

which leads to following corollary.

Corollary 4. *For all $m \geq 1$,*

$$\frac{1}{m+2}\left(1+\sum_{k=0}^{m} B_k\right)=\frac{1}{m+1}+\sum_{k=1}^{m-1}\sum_{j=1}^{m-k}\frac{(-1)^{j-1}\binom{m-k+1}{j}B_{k+j}}{(m-k+1)\binom{k+j}{k}}.$$

3. The fucntion $\tilde{\beta}_m$

In this section, we consider the function $\tilde{\beta}_m$ on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\tilde{\beta}_m(x)$ is $\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi inx}$, where

$$B_n^{(m)} = \int_0^1 \beta_m(\langle x\rangle)e^{-2\pi inx}dx = \int_0^1 \beta_m(x)e^{-2\pi inx}dx.$$

To proceed further, we note first the following lemma.

Lemma 5. *For all $m \geq 1$, $\frac{d}{dx}\beta_m(x) = 2\beta_{m-1}(x)$.*

Proof. By the definitions, we have

$$\begin{aligned}\frac{d}{dx}\beta_m(x) &= \sum_{k=0}^{m}\left(\frac{k}{k!(m-k)!}B_{k-1}(x)x^{m-k} + \frac{m-k}{k!(m-k)!}B_k(x)x^{m-k-1}\right)\\ &= \sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}B_{k-1}(x)x^{m-k} + \sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}B_k(x)x^{m-1-k}\\ &= 2\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}B_k(x)x^{m-1-k} = 2\beta_{m-1}(x),\end{aligned}$$

as claimed. □

By Lemma 5, we have that $\int_0^1 \beta_m(x)dx = \frac{\beta_{m+1}(x)}{2}\Big|_{x=0}^{x=1} = \frac{\beta_{m+1}(1)-\beta_{m+1}(0)}{2}$. On the other hand, $\beta_m(1) - \beta_m(0) = \frac{1}{(m-1)!} + \sum_{k=0}^{m-1}\frac{B_k}{k!(m-k)!}$. Thus,

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\left(\frac{1}{m!} + \sum_{k=0}^{m}\frac{B_k}{k!(m+1-k)!}\right). \tag{8}$$

Now, we are ready to determine the Fourier coefficients $B_n^{(m)}$. First, let us consider the case $n \neq 0$. By Lemma 5 and (8), we have

$$\begin{aligned}B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi inx}dx = \frac{1}{2\pi in}\int_0^1 \frac{d}{dx}\beta_m(x)e^{-2\pi inx}dx - \frac{1}{2\pi in}\;\beta_m(x)e^{-2\pi inx}\Big|_{x=0}^{x=1}\\ &= \frac{1}{\pi in}\int_0^1 \beta_{m-1}(x)e^{-2\pi inx}dx - \frac{1}{2\pi in}(\beta_m(1)-\beta_m(0))\\ &= \frac{1}{\pi in}B_n^{(m-1)} - \frac{1}{2\pi in}\left(\frac{1}{(m-1)!} + \sum_{k=0}^{m-1}\frac{B_k}{k!(m-k)!}\right).\end{aligned}$$

Hence, by induction on m, we obtain

$$B_n^{(m)} = \frac{1}{(\pi in)^{m-1}}B_n^{(1)} - \sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\left(\frac{1}{(m-j)!} + \sum_{k=0}^{m-j}\frac{B_k}{k!(m-j-k+1)!}\right).$$

On the other hand, $B_n^{(1)} = \int_0^1 \beta_1(x)e^{-2\pi inx}dx = \int_0^1 (2x-1/2)e^{-2\pi inx}dx = \frac{-1}{\pi in}$. So,

$$B_n^{(m)} = -\frac{1}{(\pi in)^m} - \sum_{j=1}^{m-1}\frac{2^{j-1}}{(2\pi in)^j}\left(\frac{1}{(m-j)!} + \sum_{k=0}^{m-j}\frac{B_k}{k!(m-j-k+1)!}\right),$$

which is equivalent to

$$B_n^{(m)} = -\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\left(\frac{1}{(m-j)!} + \sum_{k=0}^{m-j}\frac{B_k}{k!(m-j-k+1)!}\right). \tag{9}$$

The case $n = 0$ follows immediately from (8):

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\left(\frac{1}{m!} + \sum_{k=0}^{m}\frac{B_k}{k!(m+1-k)!}\right). \tag{10}$$

Let $\Omega_m = \frac{1}{(m-1)!} + \sum_{k=0}^{m-1} \frac{B_k}{k!(m-k)!}$, for all $m \geq 1$. Note that $\tilde{\beta}_m(x)$, $m \geq 1$, is piecewise C^∞. Moreover, $\tilde{\beta}_m(x)$ is continuous for those positive integers m with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega \neq 0$.

3.1. **Case $\Omega_m = 0$.** Assume first that m is a positive integer with $\Omega_m = 0$. Then $\beta_m(1) = \beta_m(0)$. So, the function $\tilde{\beta}_m(x)$ is piecewise C^∞ and continuous. Thus, the Fourier series of $\tilde{\beta}_m(x)$ converges uniformly to $\tilde{\beta}_m(x)$. So, by (9) an (10), we have

$$\tilde{\beta}_m(x) = \frac{1}{2}\left(\frac{1}{m!} + \sum_{k=0}^{m} \frac{B_k}{k!(m+1-k)!}\right) - \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\left(\frac{1}{(m-j)!} + \sum_{k=0}^{m-j} \frac{B_k}{k!(m-j-k+1)!}\right)\right) e^{2\pi i n x},$$

which implies

$$\begin{aligned}
\tilde{\beta}_m(x) &= \frac{1}{2}\left(\frac{1}{m!} + \sum_{k=0}^{m} \frac{B_k}{k!(m+1-k)!}\right) \\
&\quad + \sum_{j=1}^{m} \frac{2^{j-1}}{j!}\left(\left(\frac{1}{(m-j)!} + \sum_{k=0}^{m-j} \frac{B_k}{k!(m-j-k+1)!}\right) \sum_{n=-\infty, n\neq 0}^{\infty} \frac{-j! e^{2\pi i n x}}{(2\pi i n)^j}\right) \\
&= \frac{1}{2}\left(\frac{1}{m!} + \sum_{k=0}^{m} \frac{B_k}{k!(m+1-k)!}\right) + \sum_{j=2}^{m} \frac{2^{j-1}}{j!}\left(\left(\frac{1}{(m-j)!} + \sum_{k=0}^{m-j} \frac{B_k}{k!(m-j-k+1)!}\right)\tilde{B}_j(x)\right) \\
&\quad + \Omega_m \cdot \begin{cases} \tilde{B}_1(x), & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z} \end{cases} \\
&= \frac{1}{2}\left(\frac{1}{m!} + \sum_{k=0}^{m} \frac{B_k}{k!(m+1-k)!}\right) + \sum_{j=2}^{m} \frac{2^{j-1}}{j!}\left(\left(\frac{1}{(m-j)!} + \sum_{k=0}^{m-j} \frac{B_k}{k!(m-j-k+1)!}\right)\tilde{B}_j(x)\right) \\
&= \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1}\tilde{B}_j(x),
\end{aligned}$$

for all $x \in \mathbb{R}$. Thus, we can state the following result.

Theorem 6. *Let m be a positive integer with $\Omega_m = 0$. Then the function $\tilde{\beta}_m(x) = \sum_{k=0}^{m} \frac{\tilde{B}_k(x)}{k!(m-k)!}\langle x\rangle^{m-k}$ has the Fourier series expansion*

$$\tilde{\beta}_m(x) = \frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right) e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the convergence is uniform. Moreover,

$$\tilde{\beta}_m(x) = \sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1}\tilde{B}_j(x),$$

for all $x \in \mathbb{R}$.

3.2. **Case $\Omega_m \neq 0$.** Assume first that m is a positive integer with $\Omega_m \neq 0$. Then $\beta_m(1) \neq \beta_m(0)$. So, the function $\tilde{\beta}_m(x)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $\tilde{\beta}_m(x)$ converges pointwise to $\tilde{\beta}_m(x)$ for all $x \notin \mathbb{Z}$, and converges to

$$\frac{\beta_m(1) + \beta_m(0)}{2} = \beta_m(0) + \frac{1}{2}\Omega_m, \quad \text{for all } x \in \mathbb{Z}.$$

Then, by Theorem 6, we obtain the following result.

Theorem 7. *Let m be a positive integer with $\Omega_m \neq 0$. Then we have the following.*

$$\frac{1}{2}\Omega_{m+1} - \sum_{n=-\infty, n\neq 0}^{\infty} \left(\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x} = \begin{cases} \tilde{\beta}_m(x), & x \notin \mathbb{Z}, \\ \frac{1}{m!} B_m + \frac{1}{2}\Omega_m, & x \in \mathbb{Z}, \end{cases}$$

where the convergence is pointwise. Moreover,

$$\sum_{j=0}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} \tilde{B}_j(x) = \tilde{\beta}_m(x), \ \text{for all } x \notin \mathbb{Z},$$

and

$$\sum_{j=0, j\neq 1}^{m} \frac{2^{j-1}}{j!} \Omega_{m-j+1} \tilde{B}_j(x) = \frac{1}{m!} B_m + \frac{1}{2}\Omega_m, \ \text{for all } x \in \mathbb{Z}.$$

Theorems 6 and 7 suggest the following question: For what values of integers $m \geq 1$, does $\Omega_m = 0$ hold?

We end this section by noting that the integral $\int_0^1 \beta_m(x)dx = \sum_{k=0}^{m} \int_0^1 \frac{B_k(x)}{k!(m-k)!} x^{m-k} dx$ has been studied in [5–7] and showed that

$$\int_0^1 \beta_m(x)dx = \frac{1}{(m+1)!} + \frac{1}{(m+1)!} \sum_{k=1}^{m-1} \sum_{j=1}^{m-k} (-1)^{j-1} \binom{m+1}{k+j} B_{k+j},$$

which, by (10), gives the following identity.

Corollary 8. *For all $m \geq 1$,*

$$\frac{1}{(m+1)!} + \frac{1}{(m+1)!} \sum_{k=1}^{m-1} \sum_{j=1}^{m-k} (-1)^{j-1} \binom{m+1}{k+j} B_{k+j} = \frac{1}{2} \left(\frac{1}{m!} + \sum_{k=0}^{m} \frac{B_k}{k!(m+1-k)!} \right).$$

4. The function $\tilde{\gamma}_m$

In this section, we consider the function $\tilde{\gamma}_m$ on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\tilde{\gamma}_m(x)$ is $\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}$ with

$$C_n^{(m)} = \int_0^1 \tilde{\gamma}_m(x) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx.$$

To proceed further, we note first the following lemma.

Lemma 9. *For all $m \geq 1$, $\frac{d}{dx}\gamma_m(x) = \frac{1}{m-1}(x^{m-1} + B_{m-1}(x)) + (m-1)\gamma_{m-1}(x)$.*

Proof. By the definitions, we have

$$\begin{aligned}\frac{d}{dx}\gamma_m(x) &= \sum_{k=1}^{m-1}\left(\frac{1}{m-k}B_{k-1}(x)x^{m-k}+\frac{1}{k}B_k(x)x^{m-k-1}\right)\\ &= \sum_{k=0}^{m-2}\frac{1}{m-k-1}B_k(x)x^{m-1-k}+\sum_{k=1}^{m-1}\frac{1}{k}B_k(x)x^{m-1-k}\\ &= \frac{1}{m-1}(x^{m-1}+B_{m-1}(x))+(m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-1-k)}B_k(x)x^{m-1-k}\\ &= \frac{1}{m-1}(x^{m-1}+B_{m-1}(x))+(m-1)\gamma_{m-1}(x),\end{aligned}$$

as claimed. □

By Lemma 9, we have that $\int_0^1\gamma_m(x)dx=\frac{1}{m}(\gamma_{m+1}(x)-\frac{x^{m+1}}{m(m+1)}-\frac{1}{m(m+1)}B_{m+1}(x))\Big|_{x=0}^{x=1}$, which implies that $\int_0^1\gamma_m(x)dx=\frac{\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1+\delta_{m,0}}{m(m+1)}}{m}$. On the other hand, we have $\gamma_m(1)-\gamma_m(0)=\frac{1}{m-1}+\sum_{k=1}^{m-1}\frac{B_k}{k(m-k)}$. Thus,

$$\int_0^1\gamma_m(x)dx=\frac{1}{m}\left(\frac{1}{m+1}+\sum_{k=1}^{m}\frac{B_k}{k(m+1-k)}\right). \tag{11}$$

Define $\Lambda_m=\frac{1}{m-1}+\sum_{k=1}^{m-1}\frac{B_k}{k(m-k)}$, $\Theta_m=\sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}$, for all $m\geq 2$. Clearly, $\gamma_m(1)=\gamma_m(0)$ if and only if $\Lambda_m=0$. Now, we are ready to determine the Fourier coefficients $C_n^{(m)}$.

First, let us consider the case $n\neq 0$. By Lemma 9 and (11), we have

$$\begin{aligned}C_n^{(m)} &= \int_0^1\gamma_m(x)e^{-2\pi inx}dx=\frac{1}{2\pi in}\int_0^1\frac{d}{dx}\gamma_m(x)e^{-2\pi inx}dx-\frac{1}{2\pi in}\,\gamma_m(x)e^{-2\pi inx}\Big|_{x=0}^{x=1}\\ &= \frac{m-1}{2\pi in}\int_0^1\gamma_{m-1}(x)e^{-2\pi inx}dx-\frac{1}{2\pi in}(\gamma_m(1)-\gamma_m(0))\\ &\quad+\frac{1}{2\pi in(m-1)}\int_0^1x^{m-1}e^{-2\pi inx}dx+\frac{1}{2\pi in(m-1)}\int_0^1B_{m-1}(x)e^{-2\pi inx}dx.\end{aligned}$$

One shows that, for all $\ell\geq 1$,

$$\int_0^1x^\ell e^{-2\pi inx}dx=\begin{cases}-\sum_{k=1}^{\ell}\frac{(\ell)_{k-1}}{(2\pi in)^k}, & n\neq 0,\\ \frac{1}{\ell+1}, & n=0\end{cases}=\begin{cases}-\Theta_{\ell+1}, & n\neq 0,\\ \frac{1}{\ell+1}, & n=0\end{cases}$$

and

$$\int_0^1B_\ell(x)e^{-2\pi inx}dx=\begin{cases}-\frac{\ell!}{(2\pi in)^\ell}, & n\neq 0,\\ 0, & n=0.\end{cases}$$

Thus,

$$C_n^{(m)}=\frac{m-1}{2\pi in}C_n^{(m-1)}-\frac{1}{2\pi in}\Lambda_m-\frac{1}{2\pi in(m-1)}\Theta_m-\frac{1}{2\pi in(m-1)}\frac{(m-1)!}{(2\pi in)^{m-1}}.$$

Hence, by induction on m, we obtain

$$C_n^{(m)} = \frac{(m-1)!}{(\pi in)^{m-2}} C_n^{(2)} - \sum_{j=1}^{m-2} \frac{(m-1)_{j-1}}{(2\pi in)^j} \Lambda_{m-j+1} - \sum_{j=1}^{m-2} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Theta_{m-j+1}$$
$$- \frac{(m-1)!}{(2\pi in)^m} \sum_{k=1}^{m-2} \frac{1}{m-k}.$$

On the other hand, by direct calculations, we have that

$$C_n^{(2)} = \int_0^1 \gamma_2(x) e^{-2\pi inx} dx = \int_0^1 (x^2 - \frac{1}{2}x) e^{-2\pi inx} dx = \frac{-1}{4\pi in} - \frac{2}{(2\pi in)^2}.$$

Therefore,

$$C_n^{(m)} = -\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j} \Lambda_{m-j+1} - \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Theta_{m-j+1} - \frac{(m-1)!}{(2\pi in)^m} H_{m-1},$$

where $H_m = \sum_{j=1}^m \frac{1}{j}$ is the m-th harmonic number. Before we proceeding further, we note that

$$\begin{aligned}
\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \Theta_{m-j+1} &= \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \sum_{k=1}^{m-j} \frac{(m-j)_{k-1}}{(2\pi in)^k} \\
&= \sum_{j=1}^{m-1} \sum_{k=1}^{m-j} \frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k} (m-j)} = \sum_{s=2}^{m} \frac{(m-1)_{s-2}}{(2\pi in)^s} \sum_{j=1}^{s-1} \frac{1}{m-j} \\
&= \frac{1}{m} \sum_{s=2}^{m} \frac{(m)_s}{(2\pi in)^s} \frac{H_{m-1} - H_{m-s}}{m-s+1},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j} \Lambda_{m-j+1} &= \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j} \left(\frac{1}{m-j} + \sum_{k=1}^{m-j} \frac{B_k}{k(m-j-k+1)} \right) \\
&= \sum_{j=1}^{m-1} \sum_{k=1}^{m-j} \frac{(m-1)_{j-1} B_k}{(2\pi in)^j k(m-j-k+1)} + \sum_{j=1}^{m-1} \frac{(m-1)_{j-1}}{(2\pi in)^j (m-j)} \\
&= \frac{1}{m} \sum_{j=1}^{m-1} \sum_{k=1}^{m-j} \frac{(m)_j B_k}{(2\pi in)^j k(m-j-k+1)} + \frac{1}{m} \sum_{j=1}^{m-1} \frac{(m)_j}{(2\pi in)^j (m-j)} \\
&= \frac{1}{m} \sum_{s=1}^{m-1} \sum_{k=s}^{m-1} \frac{(m)_s B_{k-s+1}}{(2\pi in)^s (k-s+1)(m-k)} + \frac{1}{m} \sum_{s=1}^{m-1} \frac{(m)_s}{(2\pi in)^s (m-s)}
\end{aligned}$$

Hence, the coefficients $C_n^{(m)}$ are given by

$$
\begin{aligned}
C_n^{(m)} &= -\frac{1}{m}\sum_{s=2}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}-\frac{(m-1)!}{(2\pi in)^m}H_{m-1} \\
&\quad -\frac{1}{m}\sum_{s=1}^{m-1}\sum_{k=s}^{m-1}\frac{(m)_s B_{k-s+1}}{(2\pi in)^s(k-s+1)(m-k)}-\frac{1}{m}\sum_{s=1}^{m-1}\frac{(m)_s}{(2\pi in)^s(m-s)} \\
&= -\frac{1}{m}\sum_{s=1}^{m-1}\left\{\frac{H_{m-1}-H_{m-s}}{m-s+1}+\frac{1}{m-s}+\sum_{\ell=3}^{m-1}\frac{B_{\ell-s+1}}{(\ell-s+1)(m-\ell)}\right\}\frac{(m)_s}{(2\pi in)^s}-\frac{2(m-1)!}{(2\pi in)^m}H_{m-1}.
\end{aligned}
\tag{12}
$$

The case $n=0$ follows immediately from (11):

$$
C_0^{(m)}=\int_0^1\gamma_m(x)dx=\frac{1}{m}\left(\frac{1}{m+1}+\sum_{k=1}^{m}\frac{B_k}{k(m+1-k)}\right)=\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\right). \tag{13}
$$

Note that $\tilde{\gamma}_m(x)$, $m\geq 2$, is piecewise C^∞. Moreover, $\tilde{\gamma}_m(x)$ is continuous for those positive integers m with $\Lambda_m=0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Lambda_m\neq 0$.

4.1. **Case $\Lambda_m=0$.** Assume first that m is an integer ≥ 2 with $\Lambda_m=0$. Then $\gamma_m(1)=\gamma_m(0)$. So, the function $\tilde{\gamma}_m(x)$ is piecewise C^∞ and continuous. Thus, the Fourier series of $\tilde{\gamma}_m(x)$ converges uniformly to $\tilde{\gamma}_m(x)$. By (12) and (13), we have

$$
\tilde{\gamma}_m(x)=\frac{1}{m}\left(\frac{1}{m+1}+\sum_{k=1}^{m}\frac{B_k}{k(m+1-k)}\right)+\sum_{n=-\infty,n\neq 0}^{\infty}C_n^{(m)}e^{2\pi inx},
$$

where $C_n^{(m)}$ are given in (12). This implies

$$
\begin{aligned}
\tilde{\gamma}_m(x) &= \frac{1}{m}\left(\frac{1}{m+1}+\sum_{k=1}^{m}\frac{B_k}{k(m+1-k)}\right) \\
&\quad +\frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{j}\left(\frac{H_{m-1}-H_{m-s}}{m-s+1}+\sum_{\ell=s}^{m-1}\frac{B_{\ell-s+1}}{(\ell-s+1)(m-\ell)}+\frac{1}{m-s}\right)\sum_{n=-\infty,n\neq 0}^{\infty}\frac{-s!e^{2\pi inx}}{(2\pi in)^s} \\
&\quad +\frac{2}{m}H_{m-1}\sum_{n=-\infty,n\neq 0}^{\infty}\frac{-m!}{(2\pi in)^m}e^{2\pi inx} \\
&= \frac{1}{m}\left(\frac{1}{m+1}+\sum_{k=1}^{m}\frac{B_k}{k(m+1-k)}\right) \\
&\quad +\frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{H_{m-1}-H_{m-s}}{m-s+1}+\sum_{\ell=s}^{m-1}\frac{B_{\ell-s+1}}{(\ell-s+1)(m-\ell)}+\frac{1}{m-s}\right)\tilde{B}_s(x) \\
&\quad +\frac{2}{m}H_{m-1}\tilde{B}_m(x)+\Lambda_m\begin{cases}\tilde{B}_1(x), & x\notin\mathbb{Z},\\ 0, & x\in\mathbb{Z}.\end{cases}
\end{aligned}
$$

Thus, we can state the following result.

Theorem 10. *Let $m \geq 2$ be a positive integer with $\Lambda_m = 0$. Then the function $\tilde{\gamma}_m(x) = \sum_{k=1}^{m-1} \frac{\tilde{B}_k(x)}{k(m-k)} \langle x \rangle^{m-k}$ has the Fourier series expansion*

$$\tilde{\gamma}_m(x) = \frac{1}{m}\left(\frac{1}{m+1} + \sum_{k=1}^{m} \frac{B_k}{k(m+1-k)}\right) + \sum_{n=-\infty, n\neq 0}^{\infty} C_n^{(m)} e^{2\pi i n x},$$

where $C_n^{(m)}$ are given in (12), for all $x \in \mathbb{R}$, *and the convergence is uniform. Moreover,*

$$\tilde{\gamma}_m(x) = \frac{1}{m}\left(\frac{1}{m+1} + \sum_{k=1}^{m} \frac{B_k}{k(m+1-k)}\right)$$
$$+ \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{H_{m-1}-H_{m-s}}{m-s+1} + \sum_{\ell=s}^{m-1}\frac{B_{\ell-s+1}}{(\ell-s+1)(m-\ell)} + \frac{1}{m-s}\right)\tilde{B}_s(x) + \frac{2}{m}H_{m-1}\tilde{B}_m(x),$$

for all $x \in \mathbb{R}$.

4.2. **Case $\Lambda_m \neq 0$.** Assume first that m is a positive integer with $\Lambda_m \neq 0$. Then $\gamma_m(1) \neq \gamma_m(0)$. So, the function $\tilde{\gamma}_m(x)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus, the Fourier series of $\tilde{\gamma}_m(x)$ converges pointwise to $\tilde{\gamma}_m(x)$ for all $x \notin \mathbb{Z}$, and converges to

$$\frac{\gamma_m(1)+\gamma_m(0)}{2} = \gamma_m(0) + \frac{1}{2}\Lambda_m = \frac{1}{2}\Lambda_m, \quad \text{for all } x \in \mathbb{Z}$$

Then, by Theorem 10, we obtain the following result.

Theorem 11. *Let $m \geq 2$ be a positive integer with $\Lambda_m \neq 0$. Then*

$$\frac{1}{m}\left(\frac{1}{m+1} + \sum_{k=1}^{m} \frac{B_k}{k(m+1-k)}\right) + \sum_{n=-\infty, n\neq 0}^{\infty} C_n^{(m)} e^{2\pi i n x} = \begin{cases} \tilde{\gamma}_m(x), & x \notin \mathbb{Z}, \\ \frac{1}{2}\Lambda_m, & x \in \mathbb{Z}, \end{cases}$$

where $C_n^{(m)}$ are given in (12), *and the convergence is pointwise. Moreover,*

$$\frac{1}{m}\left(\frac{1}{m+1} + \sum_{k=1}^{m} \frac{B_k}{k(m+1-k)}\right)$$
$$+ \frac{1}{m}\sum_{s=1}^{m-1}\binom{m}{s}\left(\frac{H_{m-1}-H_{m-s}}{m-s+1} + \sum_{\ell=s}^{m-1}\frac{B_{\ell-s+1}}{(\ell-s+1)(m-\ell)} + \frac{1}{m-s}\right)\tilde{B}_s(x)$$
$$+ \frac{2}{m}H_{m-1}\tilde{B}_m(x) = \tilde{\gamma}_m(x), \text{ for all } x \notin \mathbb{Z},$$

and

$$\frac{1}{m}\left(\frac{1}{m+1} + \sum_{k=1}^{m} \frac{B_k}{k(m+1-k)}\right)$$
$$+ \frac{1}{m}\sum_{s=2}^{m-1}\binom{m}{s}\left(\frac{H_{m-1}-H_{m-s}}{m-s+1} + \sum_{\ell=s}^{m-1}\frac{B_{\ell-s+1}}{(\ell-s+1)(m-\ell)} + \frac{1}{m-s}\right)\tilde{B}_s(x)$$
$$+ \frac{2}{m}H_{m-1}\tilde{B}_m(x) = \frac{1}{2}\Lambda_m, \text{ for all } x \in \mathbb{Z}.$$

Theorems 10 and 11 suggest the following question: For what values of integers $m \geq 2$, does $\Lambda_m = 0$ hold?

We end this section by noting that the integral $\int_0^1 \gamma_m(x)dx = \sum_{k=1}^{m-1} \int_0^1 \frac{B_k(x)}{k(m-k)} x^{m-k} dx$ has been studied in [5–7] and showed that

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m(m^2-1)} \sum_{\ell=1}^{m-1} \sum_{j=1}^{m-\ell} \frac{(-1)^{j-1}\binom{m+1}{\ell+j} B_{\ell+j}}{\binom{m-2}{\ell-1}},$$

which, by (11), gives the following identity.

Corollary 12. *For all $m \geq 2$,*

$$\frac{1}{m(m^2-1)} \sum_{\ell=1}^{m-1} \sum_{j=1}^{m-\ell} \frac{(-1)^{j-1}\binom{m+1}{\ell+j} B_{\ell+j}}{\binom{m-2}{\ell-1}} = \frac{1}{m}\left(\frac{1}{m+1} + \sum_{k=1}^{m} \frac{B_k}{k(m+1-k)}\right).$$

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